NEW MATHEMATICAL FRAMEWORK FOR SPHERICAL GRAVITATIONAL COLLAPSE

ROBERTO GIAMBÒ, FABIO GIANNONI, GIULIO MAGLI, AND PAOLO PICCIONE

ABSTRACT. A theorem, giving necessary and sufficient condition for naked singularity formation in spherically symmetric non static spacetimes under hypotheses of physical acceptability, is formulated and proved. The theorem relates existence of singular null geodesics to existence of regular curves which are super-solutions of the radial null geodesic equation, and allows us to treat all the known examples of naked singularities from a unified viewpoint. New examples are also found using this approach, and perspectives are discussed.

1. INTRODUCTION

The validity of the so called Cosmic Censorship Conjecture, i.e. the idea that physically acceptable collapsing systems should always form blackholes is, as is well known, seriously put in doubt by various counterexamples, that is, exact solutions of the Einstein field equations exhibiting naked singularities.

Recently, we re-considered this problem in an effort of studying the validity of censorship, at least in spherical symmetry, with the application to the geodesic equations of techniques of non-linear o.d.e. without resorting to exact solutions of Einstein's. It is, in fact, obvious that exact solutions, despite their obvious interest, cannot give the complete answer to this problem due to the non-linearity of the field equations.

Our first results, presented recently [3], adopted the 'censorist' point of view, in that we provided a sufficient condition for black hole formation in spherical symmetry. However, the existence of such a condition does not necessarily mean that the room available for the Censor is very large (that is, that the hypotheses of the theorem are likely to be realized in generic situations). In order to face this problem, we take here the opposite viewpoint and obtain a sufficient condition for naked singularity formation (the condition, as we shall see, is also trivially shown to be necessary). The theorem allows us to give a simple, complete classification of (virtually) all the widespread zoo of naked singularities existing in the literature, and to produce new examples as well.

2. THE EXISTENCE OF NAKED SINGULARITIES

We consider a spherically symmetric collapsing object (the matter source can be any model compatible with the weak energy condition). The general, spherically symmetric, non–static line element in comoving coordinates t, r, θ, φ can be written in terms of three functions ν, λ, R of r and t only as follows:

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

A fundamental quantity is the mass function $m(r, t)$ defined in such a way that the equation $R = 2m$ spans the boundary of the *trapped region*, i.e. the region in which outgoing null rays re-converge:

$$(2.1) \qquad m(r, t) = (R/2)\left(1 - g^{\mu\nu}\partial_\mu R\, \partial_\nu R\right).$$

The curve $t_h(r)$ defined via $R(r, t_h(r)) = 2m(r, t_h(r))$ is called *apparent horizon*.

We consider here only those matter configurations, which admit a regular center, and we always suppose that the collapse starts from regular initial data on a Cauchy surface

($t = 0$, say), so that the singularities forming are a genuine outcome of the dynamics. If the solution is initially regular, the spacetime can become singular whenever $R = 0$ (focusing singularities) or $R' = 0$ (crossing singularities). We consider here only focusing singularities.

The locus of the zeroes of the function $R(r, t)$ defines the *singularity curve* $t_s(r)$ by the relation $R(r, t_s(r)) = 0$. Physically, $t_s(r)$ is the comoving time at which the shell of matter labeled by r becomes singular. The singularity forming at $r = 0, t = t_s(0)$ is called *central* as opposed to those occurring at $r = r_0 > 0, t = t_s(r_0)$. A singularity cannot be naked if it occurs after the formation of the apparent horizon. Since R vanishes at a singularity, any naked singularity in spherical symmetry must be massless; at a massless singularity the horizon and the singularity form simultaneously. Since regularity of the center up to singularity formation requires $m(0, t) = 0 \, \forall t < t_s(0)$, the center is always a candidate for nakedness. Usually, all other points ("non-central points") of the singularity curve lie after the formation of the apparent horizon, i.e. $t_h(r) < t_s(r)$ for $r > 0$, and the singularity is therefore covered. This always happens, for instance, if the radial pressure is positive. Then we concentrate here on the central singularity, although our results can easily be extended to non-central singularities as well.

To analyze the causal structure, observe that, if the singularity is visible to nearby observers, at least one outgoing null geodesic must exist, that meets the singularity in the past. Such a geodesic will be a solution of

$$(2.2) \qquad \begin{cases} \frac{\mathrm{d}t}{\mathrm{d}r} = \varphi(r, t), \\ t(0) = t_s(0), \end{cases}$$

where $\varphi(r, t) := e^{\lambda - \nu}$. For a problem of this kind, in which the initial point is singular (the function φ is not defined at $(0, t_s(0))$) no general results of existence/non existence are known.

In what follows we ere going to consider *sub* and *super* solutions of (2.2). We recall that a function $y_0(r)$ is called a subsolution (respectively supersolution) of an ordinary differential equation of the kind $y' = f(r, y)$ if it satisfies $y_0' \leq f(r, y_0)$ (respectively $\geq$). In [3] we have shown the following results:

2.1. Lemma. *If the weak energy condition holds and the radial stress is non-negative then the apparent horizon $t_h(r)$ is a subsolution of* (2.2).

2.2. Theorem. *If the weak energy condition holds and $\partial \varphi / \partial t < 0$ in a neighborhood of $(0, t_s(0))$ the singularity is covered.*

2.3. *Remark.* The above results hold also in presence of radial tensions (i.e. negative radial stresses) provided that p_r satisfies, near the centre, the bound $8\pi R^2 p_r \geq -1$.

We take now the opposite point of view: we search for conditions for naked singularity formation.

2.4. Definition. A curve $t_+(r)$ is called a *sub-horizon supersolution (SHS)* if

$$(2.3) \qquad t_+(0) = t_h(0), \quad t_+(r) < t_h(r) \quad t_+'(r) \geq \phi(r, t_+(r)) \, \forall r > 0.$$

2.5. Theorem. *The singularity is naked if and only if a SHS exists..*

Proof. If the singularity is naked the singular geodesic (the solution) is also a supersolution. We now proof sufficiency. Take a point (r_0, t_0) in the region $S = \{(r, t) : r > 0, t_+(r) < t < t_h(r)\}$. At this point the (regular) Cauchy problem for φ admits a unique local solution $t_g(r)$. Now the extension of this solution in the past cannot escape from S since either it would cross the supersolution from above (leading to a regular geodesic, eventually escaping from the central point at a time prior to singularity formation) or it would cross the subsolution from below. Thus it must extend back to the singularity with $\lim_{r \to 0^+} t_g(r) = t_s(0)$. $\qquad \square$

2.6. *Remark.* It is easy to show that, if a SHS exists, then $\frac{\partial \varphi}{\partial t}$ cannot be strictly negative in a neighborhood of $(0, t_0)$, in accordance with theorem 2.2.

The above analysis is limited to *radial* null geodesics. However, it can be shown that, if no radial null geodesic escapes from the singularity, then no null geodesic escapes at all. In other words, we show that a radially censored singularity is censored [4]. In the special case of dust spacetimes, this result has been was first given by Nolan and Mena [14].

2.7. **Theorem.** *A radially censored central singularity is censored.*

Proof. Suppose, by contradiction, that the center is radially censored but that there exist a singular non radial null geodesics ($\tilde{t}(r)$, say). This curve satisfies to

$$(2.4) \qquad \frac{d\tilde{t}}{dr} = \sqrt{e^{2\lambda - 2\nu} + e^{-2\nu}L^2/R^2} \geq \varphi(r, \tilde{t}),$$

where L^2 is the conserved angular momentum. Thus $\tilde{t}(r)$ is a supersolution of the null radial geodesic equation. Obviously, $\tilde{t}(r) < t_h(r)$ so that $\tilde{t}(r)$ is a SHS. As a consequence of Theorem 2.5, there exist a singular null *radial* geodesic i.e. a contradiction. $\qquad\square$

3. OLD AND NEW EXAMPLES

Comoving coordinates r, t are extremely useful in dealing with gravitational collapse because of the transparent physical meaning of the comoving time. We are, however, going to use in the present section another system of coordinates, the so-called area-radius coordinates, which were first introduced by Ori [15] to study charged dust, and then successfully applied to other models of gravitational collapse (see e.g. [9, 13]). These coordinates prove extremely useful for technical purposes.

In area-radius coordinates the comoving time is replaced by R. The velocity field of the material $v^\mu = e^{-\nu}\delta_t^\mu$ transforms to $v^\lambda = e^{-\nu}\dot{R}\delta_R^\lambda$ and therefore the transformed metric, although non diagonal, is still comoving. In a recent paper, we have found the following class of solutions of the Einstein field equations in area-radius coordinates[4]:

$$(3.1) \quad ds^2 = -\left(1 - \frac{2\Psi}{R}\right)G^2 dr^2 + 2G\frac{Y}{u}dRdr - \frac{1}{u^2}dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,.$$

In the above formulae, the two functions $\Psi(r, R)$ and $Y(r, R)$ are arbitrary (positive) functions while

$$(3.2) \qquad u^2 = Y^2 + \frac{2\Psi}{R} - 1.$$

and the function G is given in terms of a quadrature:

$$(3.3) \qquad G(r, R) = \int_R^r \frac{1}{Y(r, \sigma)}\frac{\partial(1/u)}{\partial r}(r, \sigma)\, d\sigma + \frac{1}{Y(r, r)u(r, r)}.$$

The matter distribution of the source is described in terms of energy density, radial and tangential stresses as follows (a comma denotes partial derivative):

$$(3.4) \qquad \epsilon = \frac{\Psi_{,r}}{4\pi R^2 Y uG} + \frac{\Psi_{,R}}{4\pi R^2},$$

$$(3.5) \qquad p_r = -\frac{\Psi_{,R}}{4\pi R^2}, \qquad p_t = -\frac{1}{8\pi RuG}\left(\frac{\Psi_{,rR}}{Y} - \frac{\Psi_{,r}Y_{,R}}{Y^2}\right) - \frac{\Psi_{,RR}}{8\pi R}.$$

Special, very well known sub-cases of the metrics above are:

1) The dust (Tolman-Bondi) spacetimes. All stresses vanish, energy density equals the matter density, and this implies $\Psi_{,R} = 0$ and $Y = E\Psi_{,r}$.

2) The general exact solution with vanishing radial stresses. Vanishing of p_r implies $\Psi_{,R} = 0$, while Y depends, in general, also on R. The properties of these solutions have been widely discussed in [12].

We are now going to show that Theorem 2.5 allows for a complete classification of all the naked singularities contained in the metrics (3.1). Since in turn such metrics contain all[1] the special cases for which censorship has already been investigated, we also get a unified treatment of all these - quite scattered around - results. In every such cases, the nature of the endstates has been obtained after a considerable amount of joint work by several authors, sometimes developing original mathematical techniques such as the so called root equation technique formalized by Joshi and Dwidedi [11].

The arbitrary functions Ψ and Y must be chosen in such a way that the metric admits a regular center prior to singularity formation and that the weak energy condition is satisfied. It is not difficult to check that this implies constraints on the first non-vanishing terms of the two arbitrary functions near the center. It is, however, useful for technical reasons to express such constraints in an equivalent way using Ru^2 and Y^2, as follows:

$$(3.6) \qquad Ru^2 = \sum_{i+j=3} h_{ij} r^i R^j + \sum_{i+j=3+p} h_{ij} r^i R^j + \dots,$$

$$(3.7) \qquad Y^2(r,R) = 1 + f_l r^l + \sum_{\substack{i+j=q+1 \\ j>0}} k_{ij} r^i R^j + \dots.$$

where h_{30} (that we will call α hereafter) is a strictly positive quantity, $l \geq 2$ and dots denote higher orders terms.

It can now be shown that the following Taylor expansion of $G(r,0)$ near the centre follows:

$$(3.8) \qquad G(r,0) = p\,a\,r^{p-1} - br^q,$$

where, defined

$$(3.9\text{a}) \qquad P(\tau) = \sum_{i+j=3} h_{ij} \tau^j, \qquad\qquad Q(\tau) = - \sum_{i+j=3+p} h_{ij} \tau^j,$$

$$(3.9\text{b}) \qquad S(\tau) = \sum_{\substack{i+j=q+1 \\ j>0}} k_{ij}(1-\tau^j), \qquad\qquad T(\tau) = \sum_{i+j=3} i\, h_{ij} \tau^j,$$

it is

$$(3.10) \qquad a = \int_0^1 \frac{Q(\tau)\sqrt{\tau}}{2P(\tau)^{3/2}} \, \mathrm{d}\tau, \qquad b = \int_0^1 S(\tau) \frac{\sqrt{\tau}\,T(\tau)}{2P(\tau)^{3/2}} \, \mathrm{d}\tau.$$

Thus, we finally have

$$(3.11) \qquad G(r,0) = \xi r^{n-1} + \dots,$$

where $\xi > 0$ and n is a positive integer, namely the smallest between p and $q+1$, although very special cases – where the two terms balance each other and one must compute higher order terms – can conceived. In particular, the second term is related to the acceleration of the matter flow lines and vanishes if this does.

The folllowing can now be proved [4]:

3.1. **Theorem.** *In the spacetime described by the metric* (3.1)*, a SHS exists - and therefore the singularity forming at $R = r = 0$ is naked - if and only if $n = 1$, $n = 2$, or $n = 3$ and $\xi > \alpha\xi_c$ where $\xi_c = \frac{26+15\sqrt{3}}{2}$.*

In what follows, we briefly discuss the cases already known as well as some new ones.

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[1]As far as we know the unique exceptions are the Vaidia spacetimes [1], which however can be easily accommodated in the present formalism, and the case of self-similar perfect fluids, where however the field equations become ordinary differential equations and the standard techniques of dynamical systems can be applied [5].

3.1. **Dust clouds.** In dust models both Ψ and $Y =$ depend only on r. The acceleration vanishes and therefore the index $n = p$. The solution can be given in explicit form. The spectrum of endstates for these models is very well known and was first calculated in full generality by Singh and Joshi [10]. However, our unifying framework somewhat changes the way in which the physical content of this structure must be understood, so that we discuss it briefly.

Introduce the Taylor expansions

$$(3.12) \qquad \Psi = F_0 r^3 + F_{\tilde{n}} r^{3+\tilde{n}} + \dots, \quad Y^2(r) = 1 + f_2 r^2 + f_{m+2} r^{2+m} + \dots.$$

Considering first the so-called marginally bound case (this case has been recently re-analyzed *via* o.d.e. techniques in [6, 14]) $Y \equiv 1$, we have that the index $n = p = \tilde{n}$ and $\alpha = 2F_0$, $P(\tau) = 2F_0$, $Q(\tau) = -2F_n$. The central singularity is naked if $n = 1, 2$, censored for $n > 4$. In the critical case $n = 3$ the transition is governed by the parameter a (3.10):

$$a = -\int_0^1 \frac{\sqrt{\tau}\, 2F_3}{2(2F_0)^{3/2}}\, \mathrm{d}\tau = -\frac{2F_3}{3(2F_0)^{3/2}} > \frac{2F_0}{3}\xi_c.$$

In the general (non marginally bound) case, the function Y is no longer constant. As a consequence, the expansion of the initial density and velocity mix up to produce the index n and the transitional behavior. The singularity is naked if $(F_1, f_3) \neq 0$, or if $(F_1, f_3) = 0$ but $(F_2, f_4) \neq 0$, corresponding respectively to the cases $n = 1, 2$. The critical case occurs when $(F_1, f_3) = (F_2, f_4) = 0$, but $(F_3, f_5) \neq 0$. Since $\alpha = 2F_0$, $P(\tau) = 2F_0 + f_2\tau$, $Q(\tau) = -(2F_3 + f_5\tau)$, for the singularity to be naked[2] it must be

$$a = -\int_0^1 \frac{(2F_3 + f_5\tau)\sqrt{\tau}}{2(2F_0 + f_2\tau)^{3/2}}\, \mathrm{d}\tau =$$

$$= \frac{1}{\sqrt{2F_0}}\left[\Gamma(-\frac{f_2}{2F_0})\left(\frac{F_3}{F_0} - \frac{3}{2}\frac{f_5}{f_2}\right) - \frac{1}{\sqrt{1 + \frac{f_2}{2F_0}}}\left(\frac{F_3}{F_0} - \frac{f_5}{f_2}\right)\right] > \frac{2F_0}{3}\xi_c,$$

where the function Γ is defined as

$$(3.13) \qquad \Gamma(y) = \begin{cases} -\frac{\mathrm{arcsinh}\sqrt{-y}}{(-y)^{3/2}} - \frac{\sqrt{1-y}}{y}, & \text{for } y < 0, \\ \frac{2}{3}, & \text{for } y = 0, \\ \frac{\mathrm{arcsin}\sqrt{y}}{y^{3/2}} - \frac{\sqrt{1-y}}{y}, & \text{for } y > 0. \end{cases}$$

Finally, if $(F_1, f_3) = (F_2, f_4) = (F_3, f_5) = 0$ the singularity is censored.

To discuss the physical meaning of this structure we start from an interesting example given by Dwidedi and Joshi [2].

Consider a mass distribution which generates a initially homogeneous energy density $(\Psi(r) = F_0 r^3)$. Now, keeping fixed Ψ, consider different initial velocities. If the cloud starts collapsing from rest the resulting spacetime is nothing but the "paradigm" of black-hole formation, namely, the Oppenheimer-Snyder solution. However, it suffices a fifth-order Y^2, say $Y^2(r) = 1 + f_0 r^2 + f_5 r^5$, to change drastically the behavior of the system. In fact, the fifth order term generates the critical case.

What actually happens is, that the mechanism controlling existence of SHS's depends only on the value of n, *independently* from the details of its physical origin. The mechanism works as a "kibbling machine" looking at just one term of the Taylor expansion of just function near the center. It accepts any (physically sound) input and returns an output which depends on the value of an integer: we can call it a *n-machine* .

As we shall see in next example, also the equation of state plays a similar role of "just an input" for the n-machine.

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[2]In [10] a quantity called Q_3 plays the role of a, and its expression should be corrected - formula (36) in [10] - to $Q_3 = 3a\sqrt{F_0}$, with a given above.

3.2. **Vanishing radial stresses.** These solutions are characterized by a mass depending only on r, but by a function Y^2 which contains the contribution of the internal elastic energy and thus depends also on R. As a consequence, the metric cannot be given in explicit form in general (a notable exception exists however [9]).

For simplicity, we consider here only the marginally bound case. To describe the data it is convenient to consider the Taylor expansion of the mass and of the function $g \equiv Y^2 - 1$. The constraints of physical acceptability then imply [12]:

$$\Psi(r) = F_0 r^3 + F_{\tilde{n}} r^{3+\tilde{n}} + ... \, , \ g(r, R) = -\frac{\beta_k r^k (r - R)^2}{1 + \beta_k r^k (r - R)^2} + ...,$$

where $k \geq 1$. Although the solutions are accelerating, it can be easily shown that acceleration terms do not enter into the first non-vanishing term of formula (3.11). We thus need to consider the Taylor expansion of Ru^2, given to lowest orders by $2F_0 r^3 + 2F_{\tilde{n}} r^{3+\tilde{n}} - \beta_k r^k R (r^2 + R^2 - 2rR)$. Since in general the solution is known only by quadratures, when the spectrum of endstates was first calculated in full generality [8] the Taylor expansion was used under the integral using approximation techniques near the center (delicate estimates coming from Lebesgue dominated convergence theorem are required). The endstates *look like* to depend on two parameters $\tilde{n}, k$, the second of them coming exclusively from the non-trivial equation of state. However, again, if we use the n-machine of Theorem 3.1 we see that the mechanism has no interest at all in the physical source of the parameter n: a naked singularity ($n = 1, 2$) can originate from $k = 1, 2$ *or* from $\tilde{n} = 1, 2$. The critical case always occurs at $n = 3$ and what changes is only the value of the critical parameter, since it must be $a_{k\tilde{n}} > \frac{2F_0}{3}\xi_c$, where[3]

$$a_{k\tilde{n}} = \int_0^1 \frac{-2F_3 \delta_{\tilde{n}3} \tau^{1/2} + \beta_3 \delta_{k3}(\tau^{3/2} - 2\tau^{5/2} + \tau^{7/2})}{2(2F_0)^{3/2}} \, \mathrm{d}\tau =$$
$$= \frac{1}{(2F_0)^{3/2}} \left[-\frac{2}{3} F_3 \delta_{\tilde{n}3} + \frac{8}{315} \beta_3 \delta_{k3} \right],$$

3.3. **Acceleration free non-dust solutions.** In recent years, spacetimes with cosmological ("lambda") term have attracted a renewed interest both from the astrophysical point of view, since recent observations of high-redshift type Ia supernovae suggest a non-vanishing value of lambda, and from the theoretical point of view, after the proposal of the so-called Ads-Cft correspondence in string theory. The unique model of gravitational collapse with lambda term available so far is the Tolman–Bondi–de Sitter (TBdS) spacetime, describing the collapse of spherical dust (within the framework of the present paper, these solutions are obtained choosing $\Psi(r, R) = F(r) + \frac{\Lambda}{6} R^3$). The spectrum of endstates for these solutions has recently been obtained [7].

One way to construct new collapsing solutions with lambda term is to choose a function of the type $\Psi(r, R) = F(r) + M(R) R^3$, with $M(R)$ not necessarily constant as in TBdS. It is easy to check that, in order to satisfy the requirements of physical reasonableness, one must have $\Psi(r, R) = F_0 r^3 + F_{\tilde{n}} r^{3+\tilde{n}} + M_0 R^3 + M_k R^{k+3} + ...$. Then, if one among (F_1, F_2, M_1, M_2) is non zero the singularity is naked. If otherwise they all vanish, and one among (F_3, M_3) does not, we must compute the parameter a (3.10):

$$a = -\int_0^1 \frac{\sqrt{\tau}(F_3 + M_3\tau^3)}{(2F_0 + 2M_0\tau^3)^{3/2}} \, \mathrm{d}\tau = \frac{2}{3(2F_0)^{3/2}} \left[M_3 \, \Gamma(-\frac{M_0}{F_0}) - \frac{F_3 + M_3}{\sqrt{1 + \frac{M_0}{F_0}}} \right],$$

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[3]In [8] a multiplying factor is missing.

where Γ is given in (3.13). The singularity is naked if $a > \frac{2F_0}{3}\xi_c$. In all other cases the singularity is censored.[4] Once again, the physical parameters of the collapse (in this case, there is also the cosmological constant) are mixed up by the n-machine.

4. Discussion

Starting from the piooneristic work by Eardley and Smarr [16] and Christodoulou [17] on dust clouds, the number of papers containing analysis of examples of naked singularities in spherically symmetric spacetimes can, up to now, be estimated to more than one hundred. Authors have tried to make clear the connection between, on one side, the final states of the collapse and, on the other side, the choice of the initial data and of the matter models. To do this, a mathematical framework was developed (the so called root equation technique) which allows investigation of the behavior of solutions of a differential equation at a singular point. To be applicable, this framework requires explicit knowledge of the solution of the Einstein field equations in which the geodesic motion is studied: it cannot be applied if only a part of the field equations are integrated, and can be applied only with enormous technical efforts if the solution is known only by quadratures. It is anyway obvious that, due to the non-linear character of the Einstein equations one cannot have any choice of giving general answer to the censorship problem, even in spherical symmetry, using *only* exact solutions.

So motivated, we started re-considering the censorship problem in spherical symmetry in an effort of going beyond the need for exact solutions. Essentially, the idea was to *rely* as much as possible on the "differential level" - the fact that the metric satisfies to the field equations - without resorting to the fortunate case of explicit solutions. This approach leads naturally to the application of comparison techniques in o.d.e. since the physical properties of the apparent horizon have as mathematical counterpart the fact that this horizon is a sub-solution of the geodesic motion [3]. Here we presented a new framework for the investigation of censorship, based on such ideas. The framework can be easily applied when the exact solution is known only up to quadratures, and this allowed us to give a unified view of (virtually) all the examples of naked singularities which are scattered around in the literature, as well as to produce new examples. The application of such techniques allows, however, also to investigate cases in which the metric is not known at all, such as, for instance, the case of barotropic perfect fluids.

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[4]In the particular case when all the M_k vanish for $k > 0$, we recover Tolman–Bondi–de Sitter spacetime. The condition for nakedness at the transition reduces to $\dfrac{-F_3}{(2F_0)^{5/2}\sqrt{\frac{M_0}{F_0}+1}} > \frac{\xi_c}{2}$, slightly correcting a wrong value given in [7].

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI CAMERINO, ITALY
E-mail address: roberto.giambo@unicam.it, fabio.giannoni@unicam.it

DIPARTIMENTO DI MATEMATICA, POLITECNICO DI MILANO, ITALY
E-mail address: magli@mate.polimi.it

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI CAMERINO, ITALY
(ON LEAVE FROM DEPARTAMENTO DE MATEMÁTICA,
UNIVERSIDADE DE SÃO PAULO, BRAZIL)
E-mail address: paolo.piccione@unicam.it, (piccione@ime.usp.br)